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On the Collection Formulas for Positive Words

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Abstract. For any formal commutator R of a free group F, we constructively prove the existence of a logical formula $\mathcal{E}_R$ with the following properties. First, if we apply the collection process to a positive word W of the group F, then the structure of $\mathcal{E}_R$ is determined by R, and the logical values of $\mathcal{E}_R$ are determined by W and the arrangement of the collected commutators. Second, if the commutator R was collected during the collection process, then its exponent is equal to the number of elements of the set $D(R)$ that satisfy $\mathcal{E}_R$, where $D(R)$ is determined by R. We provide examples of $\mathcal{E}_R$ for some commutators R and, as a consequence, calculate their exponents for different positive words of F. In particular, an explicit collection formula is obtained for the word $(a_1 \dots a_n)^m$, $n, m \geqslant 1$, in a group with the Abelian commutator subgroup. Also, we consider the dependence of the exponent of a commutator on the arrangement of the commutators collected during the collection process.

Keywords: commutator, collection process, free group.

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Introduction

continue our research [1] on the collection process, the concept of which was introduced by P. Hall [2]. Let W be a positive word of the free group $F = F(a_1, \dots, a_n)$, $n \geqslant 2$, i.e. W does not contain inverses of $a_1, \dots, a_n$. By rearranging step by step consecutive occurrences of elements in W with use of commutators: $QR = RQ[Q, R]$, $Q, R \in F$, the collection process transforms W into the following form:

$$W = q_1^{e_1} \dots q_j^{e_j} T_j, \qquad j \geqslant 1, \tag{1}$$

where $q_1, \dots, q_j$ are commutators in $a_1, \dots, a_m$ arranged in order of increasing weights, T_j consists of commutators of weights not less than $w(q_j)$ (the weight of q_j), the exponents $e_1, \dots, e_j$ are positive integers. Further we will not impose restrictions on the arrangement of $q_1, \dots, q_j$.

Research is developing in two directions. The first one is connected with divisibility properties of the exponents $e_1, \dots, e_j$ for some words W. In [2, Theorems 3.1 and 3.2] the application of the collection process to the word $W = (a_1 a_2)^m$, $m \geqslant 1$, leads to the formula

$$(a_1 a_2)^m = q_1^{e_1} \dots q_{j(s)}^{e_{j(s)}} \pmod{\Gamma_s(F)}, \quad s \geqslant 2, \tag{2}$$

where $\Gamma_s(F)$ is the s-th term of the lower central series of F, which is defined as follows: $\Gamma_1(F) = F$, $\Gamma_k(F) = [\Gamma_{k-1}(F), F]$, $k \geqslant 2$, and the exponents of the commutators are expressed

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in the following form:

$$e_i(m) = \sum_{k=1}^{w(q_i)} c_k \binom{m}{k}, \qquad (3)$$

where non-negative integers c_k do not depend on m. This result is significant for the theory of p-groups, since the expression $e_i(p^\alpha)$ is divisible by the prime power p^α if $w(q_i) < p$. In [3, Theorem 12.3.1] the same result is obtained for the word $(a_1 \dots a_n)^m$, $n \geqslant 1$. In [4, Theorems 5.13A and 5.13B] a similar formula with divisibility properties of the exponents of the commutators is proved for W^m, where W is an arbitrary word (not necessarily positive), $m \geqslant 1$. The work [5, Lemma 4] devoted to nilpotent products of cyclic groups and also the works [6, 7] consider the word $W = a_1^{m_1} a_2^{m_2}$ (with some restrictions on $m_1, m_2 \geqslant 1$) for which some divisibility properties of the exponents of the commutators are obtained. The author's work [1] proposes an approach to studying the exponents e_j in (1) and gives generalizations of the above results using this approach.

The second direction is connected with an explicit form of the exponents e_j in the P. Hall's collection formula (2) and, as a consequence, with explicit collection formulas (2) in groups with some restrictions (solvable length, nilpotency class of the group, etc). For example, the explicit formula

$$(a_1 a_2)^m = a_1^m a_2^m [a_2, a_1]^{\binom{m}{2}}$$

is well known for a group G, where $a_1, a_2 \in G$, $[a_2, a_1] \in Z(G)$. Formula (2) and the exponent $\binom{m}{i+1}$ of the commutator $[a_2, {}_i a_1]$, $i \geqslant 1$, have been used to prove the $(p-1)$-th Engel congruence $[a_2, {}_{p-1} a_1] = 1 \pmod{\Gamma_{p+1}(G)}$ for a group G of prime exponent p, which was the key to investigation of the restricted Burnside problem for groups of prime exponent p [3, p. 327]. With use of the exponents for more complex commutators, the 14-th Engel congruence has been proved for groups of exponent 8 in [8, 9]. Also, explicit collection formulas (2) for groups with some restrictions are considered in the works [10–13]. The explicit formula (2) for a group where any commutator with more than two occurrences of a_2 is equal to 1 has been used to prove the non-regularity of the Sylow p-subgroup of the general linear group $GL_n(\mathbb{Z}_{p^m})$ for $n \geqslant (p+2)/3$ and $m \geqslant 3$, when $(p+2)/3$ is an integer [14]. This has lead to partial solution to Wehrfritz's problem [15, Question 8.3]. The exponents for several series of the commutators in (2) have been found in an explicit form in the author's work [16].

In this paper, for any formal commutator R of the group $F(a_1, \dots, a_n)$, we constructively prove the existence of a logical formula $\mathcal{E}_R$ using which one can calculate the exponent of R using information about the initial word W in the collection process and the arrangement of the collected commutators $q_1, \dots, q_j$ (Theorem 1). The formula $\mathcal{E}_R$ has the following properties. First, its structure is determined by R, and its logical values are determined by the word W and the arrangement of the collected commutators. Second, if R was collected during the collection process, then its exponent is equal to the number of elements of the set $D(R)$ that satisfy the formula $\mathcal{E}_R$, where $D(R)$ is determined by R. We provide examples of $\mathcal{E}_R$ for some commutators R (Lemmas 1, 2) and, as a consequence, calculate their exponents for different positive words (Theorem 2). In particular, an explicit collection formula is obtained for the word $(a_1 \dots a_n)^m$, $n, m \geqslant 1$, in a group with the Abelian commutator subgroup (Theorem 3). Also, we consider the dependence of the exponent of a commutator on the arrangement of the collected commutators $q_1, \dots, q_j$ (Corollary 2).

1. Basic notation

In this paper we use the concepts formally defined in Sections 2 and 3 of the article [1]. The basic properties of the collection process and examples are also given there. In this section we will briefly describe some important concepts.

The *collection process* is a construction of the sequence of words:

$$W_0 \equiv T_0, \quad W_1 \equiv q_1^{e_1} T_1, \quad W_2 \equiv q_1^{e_1} q_2^{e_2} T_2, \quad \dots, \quad W_j = q_1^{e_1} \dots q_j^{e_j} T_j, \quad \dots \tag{4}$$

by the following rules. The initial word W_0 is a positive word of the free group $F = F(a_1, \dots, a_n)$, $n \geqslant 2$. All occurrences of the letters $a_1, \dots, a_n$ (commutators of weight 1) have *labels* (integer sequences) assigned to them, and different occurrences of the same letter have pairwise different labels of the same length. Let q_j be an arbitrary commutator from the *uncollected part* T_{j-1}. The word W_j, $j \geqslant 1$, is obtained from W_{j-1} by moving step by step all the occurrences of q_j to the beginning of the word T_{j-1} with use of commutators:

$$Q(\Lambda_u) R(\Lambda_v) = R(\Lambda_v) Q(\Lambda_u) [Q, R](\Lambda_u \Lambda_v),$$

where $\Lambda_u \Lambda_v$ is the concatenation of the labels Λ_u and Λ_v.

Denote by $D(a_k)$, $k \in \overline{1, n}$, an arbitrary fixed set of integer sequences of the same length that contains all labels of the occurrences of a_k in W_0. Assume that a commutator R arose during the collection process (4), and the parenthesis-free notation of R is $(a_{i_1}, \dots, a_{i_{w(R)}})$. Then any occurrence of R has a label that belongs to the Cartesian product $D(R) = D(a_{i_1}) \times \dots \times D(a_{i_{w(R)}})$.

Suppose that some uncollected part in (4) contains an occurrence of the commutator R. The *existence condition* of the commutator R is the predicate E_R^{Λ}, $\Lambda \in D(R)$, that is equal to 1 iff there exists a word in (4) such that its uncollected part contains the occurrence $R(\Lambda)$.

Suppose that some uncollected part in (4) contains occurrences of the commutators R and Q. The *precedence condition* for the commutators R and Q is the predicate $P_{Q,R}^{\Lambda_1 \Lambda_2}$, $\Lambda_1 \Lambda_2 \in D(Q) \times D(R)$, that is equal to 1 iff there exists a word in (4) such that, in its uncollected part, $Q(\Lambda_1)$ precedes (is to the left of) $R(\Lambda_2)$.

For the exponent e_j, $j \geqslant 1$, in (4) we have

$$e_j = |\{\Lambda \in D(q_j) \mid E_{q_j}^{\Lambda} = 1\}|. \tag{5}$$

Let R_1, R_2 be formal commutators. The predicate $R_1 \prec R_2$ is equal to 1 iff there exist commutators q_i, q_j in (4) such that $q_i = R_1$, $q_j = R_2$, $i < j$, i.e., the occurrences of R_1 were collected at an earlier stage than the occurrences of R_2 in the variant of the collection process (4).

In [1, Theorem 4.6] the following recurrence relations for the existence and precedence conditions were proved. We will use these relations further.

Suppose $\{W_j \equiv q_1^{e_1} \dots q_j^{e_j} T_j\}_{j \geqslant 0}$ is an arbitrary variant of the collection process. Then the following recurrence relations hold (if the left-hand side of a relation is defined for $\{W_j\}_{j \geqslant 0}$):

$$E_{[Q_1, {}_u R_1]}^{\Lambda_0^1 \dots \Lambda_u^1} = P_{Q_1, R_1}^{\Lambda_0^1 \Lambda_1^1} \bigwedge_{k=1}^{u-1} P_{R_1, R_1}^{\Lambda_k^1 \Lambda_{k+1}^1}, \quad u \geqslant 1; \tag{6}$$

$P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\ldots\Lambda_v^2}_{[Q_1,{}_uR_1],[Q_2,{}_vR_2]}$ is equal to

$$E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}F, \quad \text{if } u+v\geqslant 1,\ R_1=R_2,\ Q_1=Q_2; \tag{7a}$$

$$E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}P^{\Lambda_0^1\Lambda_0^2}_{Q_1,Q_2}, \quad \begin{array}{l}\text{if } u+v\geqslant 1,\ R_1=R_2,\ Q_1\neq Q_2,\\ u=0\Rightarrow w(Q_1)=1,\ v=0\Rightarrow w(Q_2)=1;\end{array} \tag{7b}$$

$$E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2}_{[Q_1,{}_uR_1],Q_2}, \quad \text{if } u,v\geqslant 1,\quad R_1\prec R_2; \tag{7c}$$

$$E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}P^{\Lambda_0^1\Lambda_0^2\ldots\Lambda_v^2}_{Q_1,[Q_2,{}_vR_2]}, \quad \text{if } u,v\geqslant 1,\quad R_2\prec R_1; \tag{7d}$$

where $[Q_1,{}_uR_1]\neq Q_2$ for $u\geqslant 1$ and $[Q_2,{}_vR_2]\neq Q_1$ for $v\geqslant 1$,

$$\Lambda_0^1\in D(Q_1),\ \Lambda_0^2\in D(Q_2),\ \Lambda_1^1,\ldots,\Lambda_u^1\in D(R_1),\ \Lambda_1^2,\ldots,\Lambda_v^2\in D(R_2),$$

$$F=P^{\Lambda_0^1\Lambda_0^2}_{Q_1,Q_2}\vee(\Lambda_0^1=\Lambda_0^2)\Big((u<v)\bigwedge_{k=1}^{u}(\Lambda_k^2=\Lambda_k^1)\vee\bigvee_{k=1}^{\min\{u,v\}}P^{\Lambda_k^2\Lambda_k^1}_{R_1,R_2}\bigwedge_{h=1}^{k-1}(\Lambda_h^2=\Lambda_h^1)\Big).$$

2. Universal existence condition

Let us fix a variant of the collection process $\{W_j\}_{j\geqslant 0}$. In [1, Corollary 4.8] it was proved that using relations (6)–(7) one can express the existence condition E_R by a formula containing at most the operations conjunction and disjunction, the predicates E_{a_i}, P_{a_i,a_j} and the equality relation on $\mathbb{Z}$.

Assume that we did not use relations (7c) and (7d) during the process of expressing E_R. If we change the variant of the collection process $\{W_j\}_{j\geqslant 0}$ (change the initial word or the arrangement of the collected commutators), then the process of expressing E_R will be exactly the same. Therefore, the resulting formula (as a construction of symbols $\wedge$, $\vee$, $=$, predicate symbols E_{a_i}, P_{a_i,a_j}) is an invariant with respect to a variant of the collection process. More precisely, if R arose during some collection process $\{W_j\}_{j\geqslant 0}$, then all predicate symbols E_{a_i}, P_{a_i,a_j} in the formula are defined only by the initial word W_0, and the formula in its logical values coincides with the existence condition E_R. Besides, since equality (5) holds, the exponent of R depends, perhaps, on the choice of the initial word, but not on the arrangement of the collected commutators.

Our aim is to construct such invariant formula for any commutator R. We now allow the formula to contain a symbol $\prec$. Let us replace relations (7c) and (7d) with

$$P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2\ldots\Lambda_v^2}_{[Q_1,{}_uR_1],[Q_2,{}_vR_2]}=E^{\Lambda_0^1\ldots\Lambda_u^1}_{[Q_1,{}_uR_1]}E^{\Lambda_0^2\ldots\Lambda_v^2}_{[Q_2,{}_vR_2]}\Big((R_1\prec R_2)P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2}_{[Q_1,{}_uR_1],Q_2}\vee(R_2\prec R_1)P^{\Lambda_0^1\Lambda_0^2\ldots\Lambda_v^2}_{Q_1,[Q_2,{}_vR_2]}\Big). \tag{8}$$

If we now use relations (6), (7a), (7b), (8) to express E_R, then on each step our choice of the desired relation does not depend on the arrangement of the collected commutators. However, there is a problem: the predicate symbols $P_{[Q_1,{}_uR_1],Q_2}$ and $P_{Q_1,[Q_2,{}_vR_2]}$ are not necessarily defined simultaneously. For example, R_1 was collected earlier than R_2 (i.e. $R_1\prec R_2$) during some collection process, and we have come across the predicate $P_{[Q_1,{}_uR_1],[Q_2,{}_vR_2]}$ during the process of expressing E_R. Then relation (7c) holds, but the predicate $P_{Q_1,[Q_2,{}_vR_2]}$ from (7d) is not defined if there does not exist an uncollected part containing both occurrences of Q_1 and $[Q_2,{}_vR_2]$ (see definition of the precedence condition). Thus, we can not continue the process of expressing E_R. To overcome this problem, we introduce the following definitions.

Definition 1. *For any commutators R_1, R_2, we call the interpretation of the predicate symbols*

$$E_{R_1},\ P_{R_1,R_2},\ \prec \tag{9}$$

the standard one with respect to a variant of the collection process $\{W_i\}_{i\geqslant 0}$ if they are defined according to the definitions in Section 1 formulated for $\{W_i\}_{i\geqslant 0}$.

The predicate symbol $\prec$ admits the standard interpretation with respect to any variant of the collection process $\{W_i\}_{i\geqslant 0}$. The same can not be said about the symbols E_{R_1}, P_{R_1,R_2}. In the first case, the occurrences of R_1 might not have arisen during the collection process. In the second case, the occurrences of R_1 and R_2 might not have arisen in the same uncollected part.

Definition 2. *Suppose Δ is a formula containing at most the symbols $\wedge$, $\vee$, $=$, the predicate symbols (9). We say that the standard interpretation of the formula Δ with respect to a variant of the collection process $\{W_i\}_{i\geqslant 0}$ is given if the symbol $=$ is interpreted as equality, all predicate symbols in Δ that allow standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$ are interpreted that way, the rest symbols (they can be only E_{R_1} and P_{R_1,R_2}) are interpreted as predicates defined arbitrarily on the sets $D(R_1)$ and $D(R_1)\times D(R_2)$, respectively.*

Theorem 1. *Suppose R is a formal commutator of the free group $F(a_1,\dots,a_n)$, $n\geqslant 2$. Then there exists a formula $\mathcal{E}_R$ with the following properties:*

1. *$\mathcal{E}_R$ contains at most the operations of conjunction, disjunction, and the following predicate symbols:*
$$E_{a_i},\ P_{a_i,a_j},\ \prec,\ =,\qquad i,j\in\overline{1,n}. \tag{10}$$
2. *If occurrences of R arose during some variant of the collection process, then, for the standard interpretation of $\mathcal{E}_R$ with respect to this variant of the collection process, the following equality holds:*
$$\mathcal{E}_R^{\Lambda} = E_R^{\Lambda},\qquad \Lambda\in D(R). \tag{11}$$

Proof. Consider the system of recurrence relations (6), (7a), (7b), (8) as formal relations of predicate symbols. Fix formal commutator R.

Let a formula Δ contain at most the operations of conjunction, disjunction, the symbol $=$, the predicate symbols (9). We say that Δ has property (M) if, for any variant of the collection process $\{W_i\}_{i\geqslant 0}$ during which R arose, the equality

$$\Delta_R^{\Lambda} = E_R^{\Lambda},\qquad \Lambda\in D(R), \tag{12}$$

holds for any standard interpretation of Δ_R^{Λ} with respect to $\{W_i\}_{i\geqslant 0}$.

Let us describe inductively the process of constructing the sequence of formulas $\{{}_i\Delta_R^{\Lambda}\}_{i\geqslant 0}$: 1) ${}_0\Delta_R^{\Lambda} = E_R^{\Lambda}$; 2) the formula ${}_{i+1}\Delta_R^{\Lambda}$ is obtained from ${}_i\Delta_R^{\Lambda}$ by replacing any predicate symbol of type E_{R_1} or P_{R_1,R_2}, where $w(R_1), w(R_2)\geqslant 2$, in ${}_i\Delta_R^{\Lambda}$ with the corresponding formula according to relations (6), (7a), (7b), (8). The sequence is finite and ends with the formula satisfying statement 1 of the theorem. This fact follows from the proof of Corollary 4.8 [1].

We prove that the formulas ${}_i\Delta_R^{\Lambda}$ has property (M) by induction on i. For $i=0$ the statement is true, since ${}_0\Delta_R^{\Lambda} = E_R^{\Lambda}$ and the predicate symbol E_R^{Λ} is standardly interpreted with respect to any variant of the collection process during which the commutator R arose. Assume that ${}_i\Delta_R^{\Lambda}$ has property (M) and the formula ${}_{i+1}\Delta_R^{\Lambda}$ is obtained by replacing a predicate symbol P in ${}_i\Delta_R^{\Lambda}$ with the corresponding formula.

Let $\{W_i\}_{i\geqslant 0}$ be a variant of the collection process during which the commutator R arose, and the symbol P does not allow the standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$. It is known that the equality ${}_i\Delta_R^\Lambda = E_R^\Lambda$, $\Lambda \in D(R)$, is true for any standard interpretation of ${}_i\Delta_R^\Lambda$ with respect to $\{W_i\}_{i\geqslant 0}$, in particular, the equality holds for any interpretation of the predicate symbol P. Therefore, P can be replaced with any formula at all, and we get $E_R^\Lambda = {}_{i+1}\Delta_R^\Lambda$ for any standard interpretation of ${}_{i+1}\Delta_R^\Lambda$ with respect to $\{W_i\}_{i\geqslant 0}$.

Now let the symbol P allow the standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$. For any relation (6), (7a), (7b), if the left-hand side of the relation allows standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$, then each predicate symbol in the right-hand side has the same property. Therefore, if P is replaced with the corresponding formula using one of these relations, then we have $E_R^\Lambda = {}_{i+1}\Delta_R^\Lambda$ for any standard interpretation of ${}_{i+1}\Delta_R^\Lambda$ with respect to $\{W_i\}_{i\geqslant 0}$. It remains to consider the case when P is replaced using relation (8).

If the left-hand side of (8) allows the standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$, then the same is true for the predicate symbols

$$E_{[Q_1,{}_uR_1]},\ E_{[Q_2,{}_vR_2]},\ \prec,$$

and at least for one of the symbols

$$P_{[Q_1,{}_uR_1],Q_2},\ P_{Q_1,[Q_2,{}_vR_2]}$$

in the right-hand side of (8). If $R_1 \prec R_2$, then, first, the symbol $P^{\Lambda_0^1\ldots\Lambda_u^1\Lambda_0^2}_{[Q_1,{}_uR_1],Q_2}$ is standardly interpreted (according to (7c)) with respect to $\{W_i\}_{i\geqslant 0}$, second, the predicate $R_2 \prec R_1$ is false. Therefore, the equality $E_R^\Lambda = {}_{i+1}\Delta_R^\Lambda$ is true for any interpretation of the symbol $P_{Q_1,[Q_2,{}_vR_2]}$ at all, hence, for any standard interpretation of ${}_{i+1}\Delta_R^\Lambda$ with respect to $\{W_i\}_{i\geqslant 0}$. If $R_2 \prec R_1$, the reasoning is analogous.

Thus, it has been proved that the last element of the sequence $\{{}_i\Delta_R^\Lambda\}_{i\geqslant 0}$, which we denote by $\mathcal{E}_R^\Lambda$, has property (M). Moreover, $\mathcal{E}_R^\Lambda$ allows a single standard interpretation with respect to $\{W_i\}_{i\geqslant 0}$, since it contains at most the predicate symbols (10), which are always standardly interpreted. □

Definition 3. For any formal commutator R of the free group $F(a_1,\ldots,a_n)$, $n \geqslant 2$, we call the formula $\mathcal{E}_R$ from Theorem 1 the *universal existence condition* of the commutator R.

Corollary 1. *If a commutator R was collected during some variant of the collection process $\{W_j\}_{j\geqslant 0}$, then its exponent is equal to*

$$|\{\Lambda \in D(R) \mid \mathcal{E}_R^\Lambda = 1\}|,$$

where the universal existence condition $\mathcal{E}_R^\Lambda$ has standard interpretation with respect to $\{W_j\}_{j\geqslant 0}$.

Corollary 2. *Suppose the universal existence condition $\mathcal{E}_R$ does not contain the predicate symbols $\prec$. Let $\{W_j\}_{j\geqslant 0}$, $\{V_j\}_{j\geqslant 0}$ be two variants of the collection process with the same initial word. If R was collected during both $\{W_j\}_{j\geqslant 0}$ and $\{V_j\}_{j\geqslant 0}$, then its exponent is the same in both cases.*

3. Examples

In this section we find the universal existence condition $\mathcal{E}_R$ for several series of commutators using the proof of Theorem 1. Namely, we construct a sequence of formulas that satisfy property (M). The sequence starts with E_R and ends with a formula satisfying statement 1 of Theorem 1. As a consequence, we get the exponents of these commutators in different collection formulas in an explicit form.

Lemma 1. *For $j, i_1, \ldots, i_s \in \{1, \ldots, n\}$ and $u_1, \ldots, u_s \geqslant 1$, where $n, s \geqslant 1$, we have*

$$\mathcal{E}^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^s\ldots\Lambda_{u_s}^s}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_s}a_{i_s}]} = \bigwedge_{k=1}^{s} P^{\Lambda_0\Lambda_1^k}_{a_j,a_{i_k}} \bigwedge_{k=1}^{s}\bigwedge_{h=1}^{u_k-1} P^{\Lambda_h^k\Lambda_{h+1}^k}_{a_{i_k},a_{i_k}}. \tag{13}$$

Proof. We use induction on s. For $s = 1$ we have

$$\mathcal{E}^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1}_{[a_j,\,{}_{u_1}a_{i_1}]} = P^{\Lambda_0\Lambda_1^1}_{a_j,a_{i_1}} \bigwedge_{h=1}^{u_1-1} P^{\Lambda_h^k\Lambda_{h+1}^k}_{a_{i_1},a_{i_1}},$$

which coincides with the result of applying relation (6) to the symbol $E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1}_{[a_j,\,{}_{u_1}a_{i_1}]}$. Assume that equality (13) is true for some s. Let us prove (13) for $s+1$.

Using (6) replace $E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^{s+1}\ldots\Lambda_{u_{s+1}}^{s+1}}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_{s+1}}a_{i_{s+1}}]}$ with the formula

$$P^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^s\ldots\Lambda_{u_s}^s\Lambda_1^{s+1}}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_s}a_{i_s}],a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}.$$

Now we use (7a) if $a_j = a_{i_{s+1}}$, otherwise we use (7b), and get the same result in both cases:

$$E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^s\ldots\Lambda_{u_s}^s}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_s}a_{i_s}]} E^{\Lambda_1^{s+1}}_{a_{i_{s+1}}} P^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^{s-1}\ldots\Lambda_{u_{s-1}}^{s-1}\Lambda_1^{s+1}}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_{s-1}}a_{i_{s-1}}],a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}.$$

Continuing this line of reasoning, after a finite number of steps we get the formula

$$\bigwedge_{k=1}^{s}\left(E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^k\ldots\Lambda_{u_k}^k}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_k}a_{i_k}]} E^{\Lambda_1^{s+1}}_{a_{i_{s+1}}}\right) P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}. \tag{14}$$

Let $\{W_j\}_{j\geqslant 0}$ be a variant of the collection process during which the commutator $[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_{s+1}}a_{i_{s+1}}]$ arose. Then all predicate symbols in (14) allow standard interpretation with respect to $\{W_j\}_{j\geqslant 0}$. For this standard interpretation, we have the following equalities of predicates for any values of variables:

$$E^{\Lambda_1^{s+1}}_{a_{i_{s+1}}} P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}} = P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}}, \qquad \bigwedge_{k=1}^{s} E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^k\ldots\Lambda_{u_k}^k}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_k}a_{i_k}]} = E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^s\ldots\Lambda_{u_s}^s}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_s}a_{i_s}]}.$$

We apply this equalities to (14) and get

$$E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^s\ldots\Lambda_{u_s}^s}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_s}a_{i_s}]} P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}.$$

Since (14) has property (M) and the reasoning above is carried out for the arbitrary variant of the collection process $\{W_j\}_{j\geqslant 0}$, then the obtained formula has property (M).

Further we should start the process of expressing the symbol $E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^s\ldots\Lambda_{u_s}^s}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_s}a_{i_s}]}$. However, by definition of the universal existence condition, the formula $\mathcal{E}^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^s\ldots\Lambda_{u_s}^s}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_s}a_{i_s}]}$ with standard interpretation is equal to the predicate $E^{\Lambda_0\Lambda_1^1\ldots\Lambda_{u_1}^1\ldots\Lambda_1^s\ldots\Lambda_{u_s}^s}_{[a_j,\,{}_{u_1}a_{i_1},\ldots,\,{}_{u_s}a_{i_s}]}$. Therefore, we can use the inductive assumption and get the formula

$$\bigwedge_{k=1}^{s} P^{\Lambda_0\Lambda_1^k}_{a_j,a_{i_k}} \bigwedge_{k=1}^{s}\bigwedge_{h=1}^{u_k-1} P^{\Lambda_h^k\Lambda_{h+1}^k}_{a_{i_k},a_{i_k}} P^{\Lambda_0\Lambda_1^{s+1}}_{a_j,a_{i_{s+1}}} \bigwedge_{h=1}^{u_{s+1}-1} P^{\Lambda_h^{s+1}\Lambda_{h+1}^{s+1}}_{a_{i_{s+1}},a_{i_{s+1}}}.$$

Collecting similar terms completes the proof. □

Lemma 2. *For $s, i, j \in \{1, \dots, n\}$, $i \neq j$, and $u, v \geqslant 1$, where $n \geqslant 1$, we have*

$$\mathcal{E}^{\Lambda_0^1\Lambda_1^1\dots\Lambda_u^1\Lambda_0^2\Lambda_1^2\dots\Lambda_v^2}_{[[a_s,{}_u a_i],[a_s,{}_v a_j]]} = P^{\Lambda_0^1\Lambda_1^1}_{a_s,a_i} \bigwedge_{k=1}^{u-1} P^{\Lambda_k^1\Lambda_{k+1}^1}_{a_i,a_i} P^{\Lambda_0^2\Lambda_1^2}_{a_s,a_j} \bigwedge_{k=1}^{v-1} P^{\Lambda_k^2\Lambda_{k+1}^2}_{a_i,a_i} (P^{\Lambda_0^1\Lambda_0^2}_{a_s,a_s} \vee (a_j \prec a_i)(\Lambda_0^1 = \Lambda_0^2)).$$

Proof. We construct the sequence of formulas according to the proof of Theorem 1 starting with

$$E^{\Lambda_0^1\Lambda_1^1\dots\Lambda_u^1\Lambda_0^2\Lambda_1^2\dots\Lambda_v^2}_{[[a_s,{}_u a_i],[a_s,{}_v a_j]]}.$$

Use relation (6):

$$P^{\Lambda_0^1\Lambda_1^1\dots\Lambda_u^1\Lambda_0^2\Lambda_1^2\dots\Lambda_v^2}_{[a_s,{}_u a_i],[a_s,{}_v a_j]}.$$

Since $i \neq j$, use relation (8):

$$E^{\Lambda_0^1\Lambda_1^1\dots\Lambda_u^1}_{[a_s,{}_u a_i]} E^{\Lambda_0^2\Lambda_1^2\dots\Lambda_v^2}_{[a_s,{}_v a_j]} \left((a_i \prec a_j) P^{\Lambda_0^1\Lambda_1^1\dots\Lambda_u^1\Lambda_0^2}_{[a_s,{}_u a_i],a_s} \vee (a_j \prec a_i) P^{\Lambda_0^1\Lambda_0^2\Lambda_1^2\dots\Lambda_v^2}_{a_s,[a_s,{}_v a_j]}\right).$$

Next we use (6) and (7a) twice:

$$P^{\Lambda_0^1\Lambda_1^1}_{a_s,a_i} \bigwedge_{k=1}^{u-1} P^{\Lambda_k^1\Lambda_{k+1}^1}_{a_i,a_i} P^{\Lambda_0^2\Lambda_1^2}_{a_s,a_j} \bigwedge_{k=1}^{v-1} P^{\Lambda_k^2\Lambda_{k+1}^2}_{a_i,a_i} \wedge \left((a_i \prec a_j) P^{\Lambda_0^1\Lambda_0^2}_{a_s,a_s} \vee (a_j \prec a_i)(P^{\Lambda_0^1\Lambda_0^2}_{a_s,a_s} \vee (\Lambda_0^1 = \Lambda_0^2))\right).$$

Now we simplify the expression in brackets using logical transformations and the fact that the expression $(a_j \prec a_i) \vee (a_i \prec a_j)$ with standard interpretation is true for any variant of the collection process during which the commutator $[[a_s,{}_u a_i],[a_s,{}_v a_j]]$ arose. □

Theorem 2. *Suppose a formal commutator R was collected during some variant of the collection process $\{W_j\}_{j\geqslant 0}$ and its exponent is equal to $e(R)$. The following statements hold.*

1. *If $W_0 \equiv (a_1 \dots a_n)^m$, $n, m \geqslant 1$, and $R = [a_j,{}_{u_1} a_{i_1}, \dots,{}_{u_s} a_{i_s}]$, then*

$$e(R) = \sum_{\lambda_0=0}^{m-1} \prod_{\substack{k=1,\dots,s;\\ j<i_k}} \binom{\lambda_0+1}{u_k} \prod_{\substack{k=1,\dots,s;\\ j\geqslant i_k}} \binom{\lambda_0}{u_k}.$$

2. *If $W_0 \equiv (a_1 \dots a_n)^m$, $n, m \geqslant 1$, and $R = [[a_s,{}_u a_i],[a_s,{}_v a_j]]$, $i \neq j$, then*

$$e(R) = \sum_{\lambda_0^1=1}^{m+\delta_{(a_j\prec a_i)}-1} \binom{\lambda_0^1 - \delta_{(a_j\prec a_i)} + \delta_{(s<i)}}{u} \binom{\lambda_0^1 + \delta_{(s<j)}}{v+1},$$

   where $\delta_A = 1$ if the proposition A is true, otherwise $\delta_A = 0$.

3. *If $W_0 \equiv a_1^{m_1} \dots a_n^{m_n}$, $n, m_1, \dots, m_n \geqslant 1$, and $R = [a_j,{}_{u_1} a_{i_1}, \dots,{}_{u_s} a_{i_s}]$, then*

$$e(R) = \binom{m_j}{u+1} \prod_{k=1,\dots,s} \binom{m_{i_k}}{u_k},$$

   where $u = u_l$ if there exists $i_l = j$, otherwise $u = 0$.

Proof. Consider a variant of the collection process with the initial word

$$W_0 \equiv a_1(1) \dots a_n(1) \dots a_1(m) \dots a_n(m).$$

We have

$$P_{a_i,a_j}^{(\lambda_1,\lambda_2)} = (\lambda_1 < \lambda_2) \vee (\lambda_1 = \lambda_2)(i < j), \quad \lambda_1, \lambda_2 \in \{1, \dots, m\},\ i, j \in \overline{1, n}.$$

Assume that the commutator $[a_j,\, {}_{u_1}a_{i_1}, \dots,\, {}_{u_s}a_{i_s}]$ arose during the collection process. From Lemma 1 is follows that

$$E_{[a_j,\, {}_{u_1}a_{i_1},\dots,\, {}_{u_s}a_{i_s}]}^{(\lambda_0,\lambda_1^1,\dots,\lambda_{u_1}^1,\dots,\lambda_1^s,\dots,\lambda_{u_s}^s)} = \bigwedge_{k=1}^{s} ((\lambda_0 < \lambda_1^k) \vee (\lambda_0 = \lambda_1^k)(j < i_k)) \bigwedge_{k=1}^{s} \bigwedge_{h=1}^{u_k-1} (\lambda_h^k < \lambda_{h+1}^k),$$

where $\lambda_0, \lambda_1^1, \dots, \lambda_{u_1}^1, \dots, \lambda_1^s, \dots, \lambda_{u_s}^s \in \{1, \dots, m\}$. Then the exponent of this commutator is equal to the number of solutions of the following system:

$$\begin{cases} 1 \leqslant \lambda_0 \leqslant m; \\ \lambda_0 \leqslant \lambda_1^k < \lambda_2^k < \dots < \lambda_{u_k} \leqslant m, \quad k \in \overline{1, s},\ j < i_k; \\ \lambda_0 < \lambda_1^k < \lambda_2^k < \dots < \lambda_{u_k} \leqslant m, \quad k \in \overline{1, s},\ j \geqslant i_k. \end{cases}$$

Taking into account that the number of integer sequence $(x_1, \dots, x_m)$ that satisfy the condition $1 \leqslant x_1 \leqslant \dots \leqslant x_m \leqslant n$ is equal to $\binom{n}{m}$, we get the number of solutions:

$$\sum_{\lambda_0=1}^{m} \prod_{\substack{k=1,\dots,s;\\ j<i_k}} \binom{m-\lambda_0+1}{u_k} \prod_{\substack{k=1,\dots,s;\\ j\geqslant i_k}} \binom{m-\lambda_0}{u_k} = \sum_{\lambda_0=0}^{m-1} \prod_{\substack{k=1,\dots,s;\\ j<i_k}} \binom{\lambda_0+1}{u_k} \prod_{\substack{k=1,\dots,s;\\ j\geqslant i_k}} \binom{\lambda_0}{u_k}.$$

Now assume that the commutator $[[a_s,\, {}_u a_i], [a_s,\, {}_v a_j]]$ for $u, v \geqslant 1$, $i \neq j$ arose during the collection process. Then by Lemma 2 we have

$$E_{[[a_s,\, {}_u a_i],[a_s,\, {}_v a_j]]}^{(\lambda_0^1,\lambda_1^1,\dots,\lambda_u^1,\lambda_0^2,\lambda_1^2,\dots,\lambda_v^2)} = \big((\lambda_0^1 < \lambda_1^1) \vee (\lambda_0^1 = \lambda_1^1)(s < i)\big)\big((\lambda_0^2 < \lambda_1^2) \vee (\lambda_0^2 = \lambda_1^2)(s < j)\big) \wedge$$
$$\wedge \bigwedge_{k=1}^{u-1} (\lambda_k^1 < \lambda_{k+1}^1) \bigwedge_{k=1}^{v-1} (\lambda_k^2 < \lambda_{k+1}^2)((\lambda_0^1 < \lambda_0^2) \vee (a_j \prec a_i)(\lambda_0^1 = \lambda_0^2)).$$

Therefore, the exponent of $[[a_s,\, {}_u a_i], [a_s,\, {}_v a_j]]$ is equal to the number of solutions of the following system:

$$\begin{cases} 1 \leqslant \lambda_0^1 \leqslant m; \\ 1 \leqslant \lambda_0^2 \leqslant m; \\ \lambda_0^1 - \delta_{(a_j \prec a_i)} + 1 \leqslant \lambda_0^2; \\ \lambda_0^1 - \delta_{(s<i)} + 1 \leqslant \lambda_1^1 < \lambda_2^1 < \dots < \lambda_u^1 \leqslant m; \\ \lambda_0^2 - \delta_{(s<j)} + 1 \leqslant \lambda_1^2 < \lambda_2^2 < \dots < \lambda_v^2 \leqslant m. \end{cases}$$

We get the following expression:

$$\sum_{\lambda_0^1=1}^{m} \sum_{\lambda_0^2=\lambda_0^1-\delta_{(a_j \prec a_i)}+1}^{m} \binom{m-\lambda_0^1+\delta_{(s<i)}}{u} \binom{m-\lambda_0^2+\delta_{(s<j)}}{v} =$$
$$= \sum_{\lambda_0^1=1}^{m} \sum_{\lambda_0^2=\delta_{(s<j)}}^{m-\lambda_0^1+\delta_{(a_j \prec a_i)}+\delta_{(s<j)}-1} \binom{m-\lambda_0^1+\delta_{(s<i)}}{u} \binom{\lambda_0^2}{v} =$$

since $0 \leqslant \delta_{(s<j)} \leqslant 1$ and $v \geqslant 1$, we change the lower limit of λ_0^2 to v and apply a well-known summation formula:

$$= \sum_{\lambda_0^1=1}^{m+\delta_{(a_j \prec a_i)}-1} \binom{m-\lambda_0^1+\delta_{(s<i)}}{u} \binom{m-\lambda_0^1+\delta_{(a_j \prec a_i)}+\delta_{(s<j)}}{v+1} =$$

change the order of summation:

$$= \sum_{\lambda_0^1=1-\delta_{(a_j \prec a_i)}}^{m-1} \binom{\lambda_0^1+\delta_{(s<i)}}{u} \binom{\lambda_0^1+\delta_{(a_j \prec a_i)}+\delta_{(s<j)}}{v+1} =$$

$$= \sum_{\lambda_0^1=1}^{m+\delta_{(a_j \prec a_i)}-1} \binom{\lambda_0^1-\delta_{(a_j \prec a_i)}+\delta_{(s<i)}}{u} \binom{\lambda_0^1+\delta_{(s<j)}}{v+1}.$$

Now let us consider a variant of the collection process with the initial word

$$W_0 \equiv a_1(1)\ldots a_1(m_1)\ldots a_n(1)\ldots a_n(m_n).$$

We have

$$P_{a_i,a_j}^{(\lambda_1,\lambda_2)} = (\lambda_1 < \lambda_2)(i=j) \vee (i<j), \quad \lambda_1,\lambda_2 \in \{1,\ldots,m\},\ i,j \in \overline{1,n}.$$

Assume that the commutator $[a_j,\, _{u_1}a_{i_1},\ldots,\, _{u_s}a_{i_s}]$ arose during the collection process. From Lemma 1 it follows that

$$E_{[a_j,\, _{u_1}a_{i_1},\ldots,\, _{u_s}a_{i_s}]}^{(\lambda_0,\lambda_1^1,\ldots,\lambda_{u_1}^1,\ldots,\lambda_1^s,\ldots,\lambda_{u_s}^s)} = \bigwedge_{k=1}^{s} ((\lambda_0 < \lambda_1^k)(j=i_k) \vee (j<i_k)) \bigwedge_{k=1}^{s} \bigwedge_{h=1}^{u_k-1} (\lambda_h^k < \lambda_{h+1}^k).$$

Then we get the following system:

$$\begin{cases} 1 \leqslant \lambda_0 \leqslant m_j; \\ \lambda_0 < \lambda_1^k < \lambda_2^k < \ldots < \lambda_{u_k} \leqslant m_{i_k}, \quad k \in \overline{1,s},\ j=i_k; \\ 1 \leqslant \lambda_1^k < \lambda_2^k < \ldots < \lambda_{u_k} \leqslant m_{i_k}, \quad k \in \overline{1,s},\ j<i_k. \end{cases}$$

The number of solutions of this system is equal to

$$\sum_{\lambda_0=1}^{m_j} \prod_{\substack{k=1,\ldots,s \\ j=i_k}} \binom{m_{i_k}-\lambda_0}{u_k} \prod_{\substack{k=1,\ldots,s; \\ j<i_k}} \binom{m_{i_k}}{u_k} = \sum_{\lambda_0=0}^{m_j-1} \prod_{\substack{k=1,\ldots,s \\ j=i_k}} \binom{\lambda_0}{u_k} \prod_{\substack{k=1,\ldots,s; \\ j<i_k}} \binom{m_{i_k}}{u_k}.$$

If none of the numbers $i_1,\ldots,i_s$ is equal to j, then we get

$$m_j \prod_{k=1,\ldots,s} \binom{m_{i_k}}{u_k}.$$

If some i_l is equal to j (in this case i_l is unique), then the exponent is equal to

$$\binom{m_j}{u_l+1} \prod_{k=1,\ldots,s} \binom{m_{i_k}}{u_k}.$$

□

Theorem 3. *Suppose G is a group with the Abelian commutator subgroup, $a_1,\ldots,a_n \in G$, $n,m \in \mathbb{N}$. Then the following formula holds:*

$$(a_1\ldots a_n)^m = a_1^m\ldots a_n^m \prod_{j=2}^{n} \prod_{(u_1,\ldots,u_n)\in M^j_{n,m}} [a_j,{}_{u_1}a_1,\ldots,{}_{u_n}a_n]^{\sum\limits_{k=0}^{m-1}\prod\limits_{s=1}^{j}\binom{k}{u_s}\prod\limits_{s=j+1}^{n}\binom{k+1}{u_s}},$$

where $M^j_{n,m} = \{(u_1,\ldots,u_n) \in \{0,\ldots,m\}^n \mid u_1+\cdots+u_n > 0;$ the first $u_i > 0$ has $i < j\}$.

Proof. Consider the word $(a_1\ldots a_n)^m$ of the free group $F(a_1,\ldots,a_n)$. Let us apply the collection process to this word. First, we collect letters in the following order: $a_1,\ldots,a_n$ and get the word

$$a_1^m\ldots a_n^m \prod [a_j,{}_{u_1}a_1,\ldots,{}_{u_n}a_n],$$

where the product is over some non-negative integers $j,u_1,\ldots,u_n$. After that we collect the commutators $[a_j,{}_{u_1}a_1,\ldots,{}_{u_n}a_n]$ in some fixed order. From Theorem 2 it follows that we get the following formula in the group G:

$$(a_1\ldots a_n)^m = a_1^m\ldots a_n^m \prod_{j\in J} \prod_{(u_1,\ldots,u_n)\in M^j_{n,m}} [a_j,{}_{u_1}a_1,\ldots,{}_{u_n}a_n]^{\sum\limits_{k=0}^{m-1}\prod\limits_{s=1}^{j}\binom{k}{u_s}\prod\limits_{s=j+1}^{n}\binom{k+1}{u_s}},$$

where it remains to find the sets J and $M^j_{n,m}$. Note that the use of Theorem 2 in the case when some $u_s = 0$ is correct, since $\binom{a}{0} = 1$ for any $a \geqslant 0$.

Obviously, $J \subseteq \{2,\ldots,n\}$, since a_1 was collected first. Further, the expression in the exponent is equal to 0 when $u_s \geqslant m+1$, therefore, we have $M^j_{n,m} \subseteq \{0,\ldots,m\}^n$. At least one element of the sequence $(u_1,\ldots,u_n) \in M^j_{n,m}$ is not equal to 0, since otherwise we get the commutator a_j. Moreover, the first $u_i > 0$ has the index $i < j$, since the commutators were collected in the order $a_1,\ldots,a_n$. Thus, the following inclusions have been proved:

$$J \subseteq \{2,\ldots,n\},$$
$$M^j_{n,m} \subseteq \{(u_1,\ldots,u_n) \in \{0,\ldots,m\}^n \mid u_1+\cdots+u_n > 0;\ \text{the first } u_i > 0 \text{ has } i < j\}.$$

To prove the reverse inclusions, we assume that the expression in the exponent of $[a_j,{}_{u_1}a_1,\ldots,{}_{u_n}a_n]$ is not equal to 0 for some sequence $(j,u_1,\ldots,u_n)$. From the proof of Theorem 2 it follows that there exist some values of the variables for which the formula

$$\bigwedge_{k=1}^{s} P^{\Lambda_0\Lambda_1^k}_{a_j,a_{i_k}} \bigwedge_{k=1}^{s} \bigwedge_{h=1}^{u_{i_k}-1} P^{\Lambda_h^k\Lambda_{h+1}^k}_{a_{i_k},a_{i_k}}$$

is equal to 1, where $1 \leqslant i_1 < \ldots < i_s \leqslant n$ and $u_{i_k} > 0$ for any k. Therefore, in the initial word $(a_1\ldots a_n)^m$, there are u_{i_1} occurrences of a_{i_1}, u_{i_2} occurrences of a_{i_2}, etc to the right of $a_j(\Lambda_0)$. Since the letters were collected in the order $a_{i_1},\ldots,a_{i_s}$, and $j \geqslant 2$, $i_1 < j$, the commutator $[a_j,{}_{u_{i_1}}a_{i_1},\ldots,{}_{u_{i_s}}a_{i_s}] = [a_j,{}_{u_1}a_1,\ldots,{}_{u_n}a_n]$ arose during the collection process. □

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О собирательных формулах для положительных слов

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Аннотация. Для любого формального коммутатора R свободной группы F мы конструктивно доказываем существование логической формулы $\mathcal{E}_R$ со следующими свойствами. Во-первых, ее строение определяется структурой R, а логические значения определяются положительным словом группы F, к которому применяется собирательный процесс, и порядком сбора коммутаторов. Во-вторых, если в ходе собирательного процесса был собран коммутатор R, то его показатель степени равен количеству элементов множества $D(R)$, удовлетворяющих $\mathcal{E}_R$, где $D(R)$ определяется структурой R. В работе приведены примеры такой формулы для разных коммутаторов, как следствие, вычислены их показатели степеней для разных положительных слов F. В частности, получена в явном виде собирательная формула для слова $(a_1 \dots a_n)^m$, $n, m \geqslant 1$ в группе с абелевым коммутантом. Рассмотрен вопрос о зависимости показателя степени коммутатора от порядка сбора коммутаторов в ходе собирательного процесса.

Ключевые слова: коммутатор, собирательный процесс, свободная группа.